Visualizing Dessins d'Enfants on the Torus

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Abstract

A graph $\Gamma = (V, E)$ is a pair consisting of vertices V and edges E. Such a graph is said to be planar if it can be embedded $\Gamma \hookrightarrow S^2(\mathbb{R})$ on the sphere such that its edges do not cross. Similarly, such a graph is said to be toroidal if it is not planar, yet can be embedded $\Gamma \hookrightarrow \mathbb{T}^2(\mathbb{R})$ on the torus such that its edges do not cross. This project seeks to determine the monodromy groups of such graphs.

In this presentation, we discuss various examples of toroidal graphs, including the Utility Graph, Petersen Graph, and more generally 3-regular graphs. Given a particular embedding $\Gamma \hookrightarrow \mathbb{T}^2(\mathbb{R})$ of a graph with $N = |E|$ edges, we can label the edges E to find a particular group $G = \operatorname{im}\left[\pi_1\left(\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}\right) \to S_N\right]$ called the monodromy group; it is the "Galois closure" of the group of automorphisms of the graph. We will discuss some of the challenges of determining the structure of these groups, and present visualizations of group actions on the torus.

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Toroidal Graphs

We say that a graph $\overline{\Gamma} = (V, E)$ is a **bipartite, toroidal graph** if

- $\overline{\Gamma}$ can be embedded in the torus $\mathbb{T}^2(\mathbb{R})$ without crossings yet cannot be embedded in sphere $S^2(\mathbb{R})$ without crossings, and
- the vertices $V = B \cup W$ are a disjoint union of "black" vertices B and "white" vertices W such that no vertex $P \in B$ ($P \in W$, respectively) is connected by an edge which lands in B again (W again, respectively).

Even if Γ is a toroidal graph which is not bipartite, we turn Γ into a bipartite graph $\overline{\Gamma} = (B \cup W, E)$ via a **subdivision**:

- color the vertices B of Γ as "black",
- color the midpoints W of the edges of Γ as "white", and
- let the edges E be the subdivided edges of Γ.

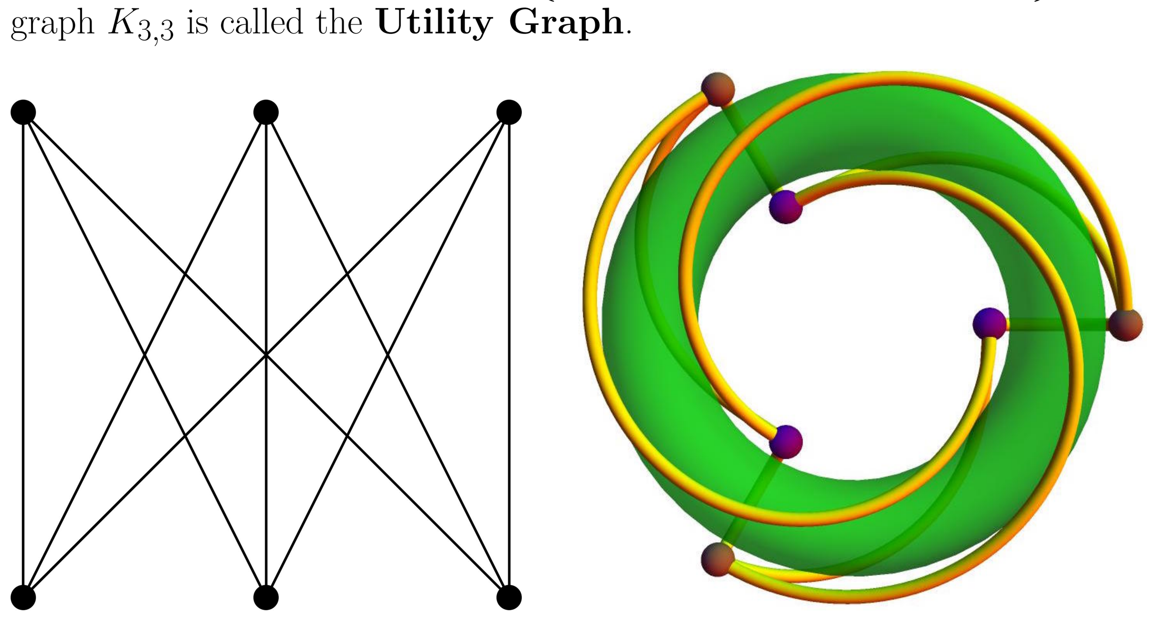
Example: Complete Bipartite Graphs

The **Complete Bipartite Graph** $K_{m,n}$ is a graph with $|V| = m + n$ vertices and $|E| = m\,n$ edges, where each of the $|B| = m$ "black" vertices is connected by an edge to each of the $|W| = n$ "white" vertices.

In 1964, Gerhard Ringel showed $K_{m,n} \hookrightarrow X$ can be embedded on a compact connected Riemann surface X of genus g without edge crossings, where $|V| - |E| + |F| = 2 - 2g$ in terms of

$$g = \left\lceil \frac{(m-2)\,(n-2) + 3}{4} \right\rceil .$$

Hence the only complete bipartite graphs $K_{m,n}$ which are toroidal graphs correspond to the pairs $(m, n) \in \{(3,3),\ (3,4),\ (3,5),\ (3,6),\ (4,4)\}$. The graph $K_{3,3}$ is called the **Utility Graph**.



Utility Graph

Degree Sequences

Let $\overline{\Gamma} = (V, E)$ be a bipartite, toroidal graph. Upon writing F as the collection of midpoints of faces of $\overline{\Gamma}$, denote the **Degree Sequence** of $\overline{\Gamma}$ as the multiset

$$\mathcal{D} = \Big\{ \{e_P \,|\, P \in B\},\ \{e_P \,|\, P \in W\},\ \{e_P \,|\, P \in F\} \Big\}$$

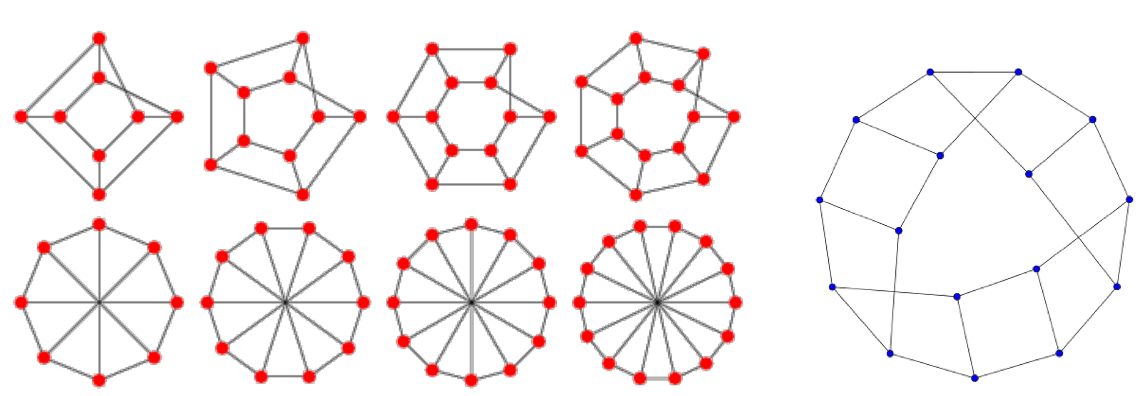
where e_P is the number of edges emanating from each "black" vertex $P \in B$ for the first component, "white vertex $P \in W$ for the second component, and e_P is the number of "white" vertices surrounding each face midpoint $P \in F$ for the third component. The number of edges of $\overline{\Gamma}$ is given by the Degree Sum Formula

$$N = \sum_{P \in B} e_P = \sum_{P \in W} e_P = \sum_{P \in F} e_P = |B| + |W| + |F| = |E|.$$

Example: The Möbius Ladder

The **Möbius Ladder** M_n is a toroidal graph. In fact, since $n = 2\,m$ must be even, the subdivision of M_n has $N = 6\,m$ edges and has the degree sequence

$$\mathcal{D} = \Bigg\{ \underbrace{\{3, 3, \ldots, 3\}}_{2\,m \text{ copies}},\ \underbrace{\{2, 2, \ldots, 2\}}_{3\,m \text{ copies}},\ \underbrace{\{4, \ldots, 4,\ 2\,m+4\}}_{(m-1) \text{ copies of } 4} \Bigg\}.$$



Möbius Ladder

Example: Toroidal Cubic Graphs

A toroidal graph Γ is said to be a **Cubic Graph** if it is 3-regular, that is, its subdivision $\overline{\Gamma}$ is a graph with a degree sequence in the form

$$\mathcal{D} = \Bigg\{ \underbrace{\{3, 3, \ldots, 3\}}_{2\,n \text{ copies}},\ \underbrace{\{2, 2, \ldots, 2\}}_{3\,n \text{ copies}},\ \{e_P \,|\, P \in F\} \Bigg\}$$

A toroidal cubic graph has $|B| = 2\,n$ vertices, $|W| = 3\,n$ edges, and $|F| = n$ faces where the set $\{e_P \,|\, P \in F\}$ is a partition of $N = 6\,n$.

In particular, if the faces are hexagonal, then its subdivision is a graph with a degree sequence in the form

$$\mathcal{D} = \Bigg\{ \underbrace{\{3, 3, \ldots, 3\}}_{2\,n \text{ copies}},\ \underbrace{\{2, 2, \ldots, 2\}}_{3\,n \text{ copies}},\ \underbrace{\{6, 6, \ldots, 6\}}_{n \text{ copies}} \Bigg\}$$

as a collection of three partitions of $N = 6\,n$.

Monodromy Group

We associate a group to $\overline{\Gamma}$ as follows. Label the edges in E from 1 through N. Since the compact, connected surface X is oriented, read off the labels counterclockwise of the edges incident to each vertex $P \in B$ ($P \in W$, respectively) to find the integers $B_{P,1},\ B_{P,2},\ \ldots,\ B_{P,e_P}$ ($W_{P,1},\ W_{P,2},\ \ldots,\ W_{P,e_P}$, respectively). Define the **Monodromy Group** for an embedding $\overline{\Gamma} \hookrightarrow \mathbb{T}^2(\mathbb{R})$ as $G = \langle \sigma_0,\ \sigma_1,\ \sigma_\infty \rangle$ in terms of the permutations

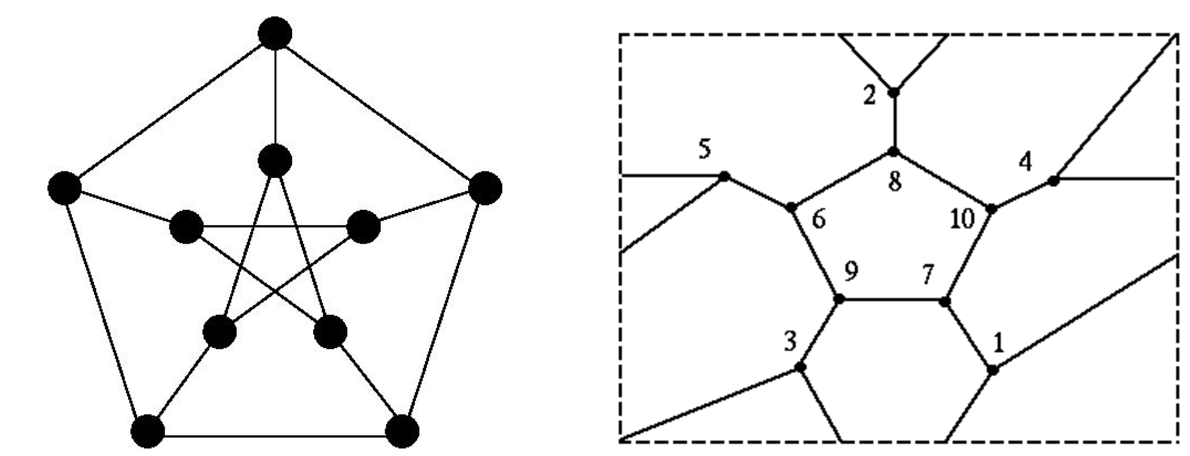
$$\sigma_0 = \prod_{P \in B} \left(B_{P,1}\ B_{P,2}\ \cdots\ B_{P,e_P}\right)$$

$$\sigma_1 = \prod_{P \in W} \left(W_{P,1}\ W_{P,2}\ \cdots\ W_{P,e_P}\right)$$

$$\sigma_\infty = \sigma_1^{-1} \circ \sigma_0^{-1}$$

This is subgroup of S_N. It is transitive if and only if $\overline{\Gamma}$ is path connected.

Example: Petersen Graph



Petersen Graph

The **Petersen graph** $G(5, 2)$ is a toroidal graph. Its subdivision is a graph with a degree sequence

$$\mathcal{D} = \Bigg\{ \underbrace{\{3, 3, \ldots, 3\}}_{10 \text{ vertices}},\ \underbrace{\{2, 2, \ldots, 2\}}_{15 \text{ edges}},\ \{5, 5, 5, 6, 9\} \Bigg\}$$

as a collection of three partitions of $N = |B| + |W| + |F| = 30$.

One monodromy group associated with this graph is $G = \langle \sigma_0,\ \sigma_1,\ \sigma_\infty \rangle$ as a transitive subgroup of S_{30} generated by

$$\sigma_0 = (1\,3\,5)\,(7\,6\,9)\,(13\,11\,10)\,(2\,29\,21)\,(4\,19\,18)\,(12\,28\,20)\,(8\,24\,25)\,(14\,15\,22)\,(16\,17\,23)\,(30\,26\,27)$$

$$\sigma_1 = (1\,2)\,(3\,4)\,(5\,6)\,(7\,8)\,(9\,10)\,(11\,12)\,(13\,14)\,(15\,16)\,(17\,18)\,(19\,20)\,(21\,22)\,(23\,24)\,(25\,26)\,(27\,28)\,(29\,30)$$

$$\sigma_\infty = (1\,6\,8\,26\,29)\,(11\,14\,21\,30\,28)\,(18\,20\,27\,25\,23)\,(4\,17\,15\,13\,9\,5)\,(2\,22\,16\,24\,7\,10\,12\,19\,3)$$

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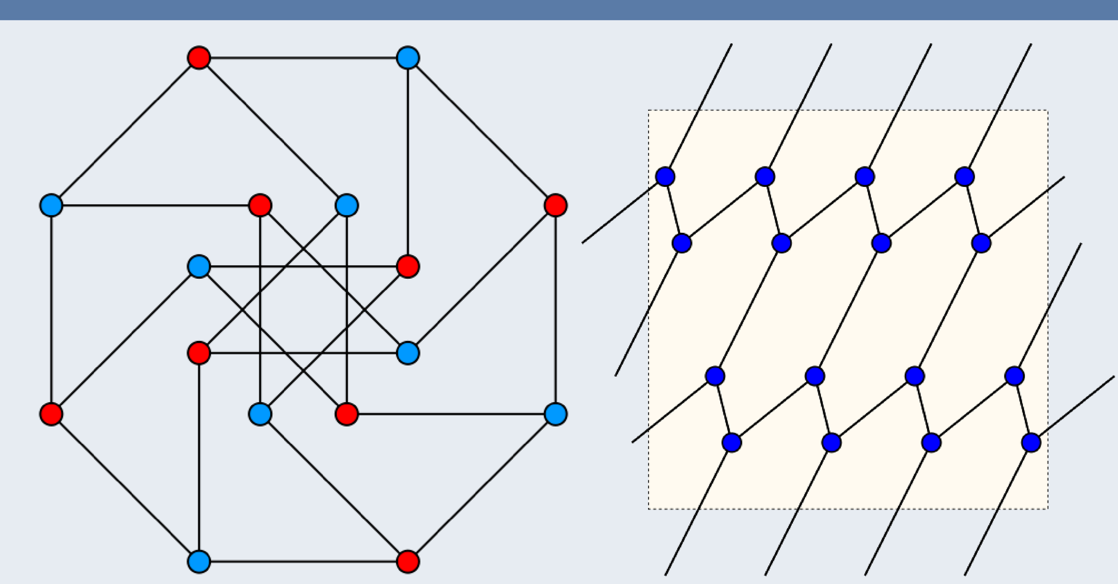
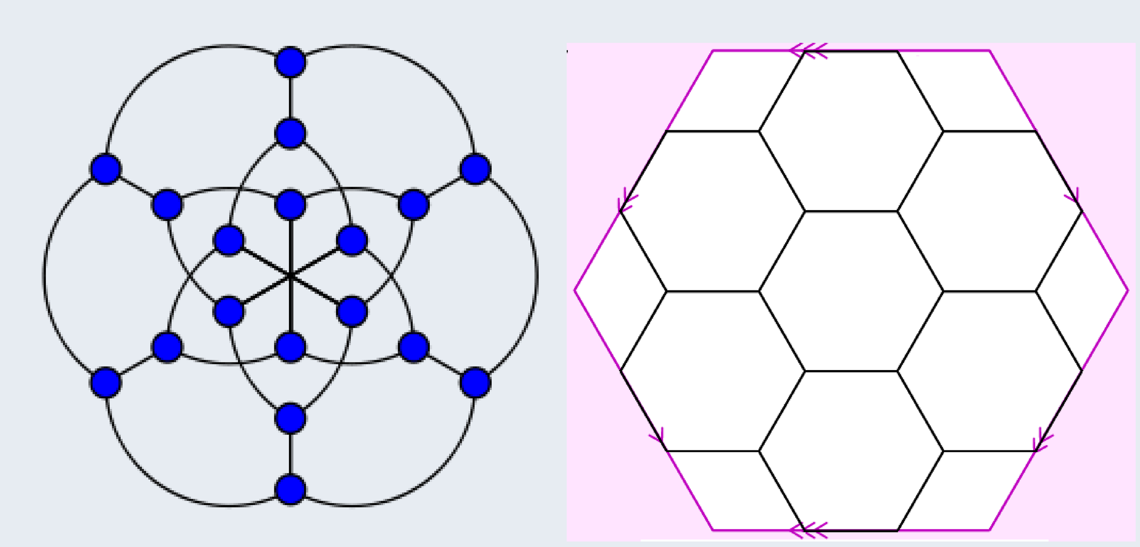
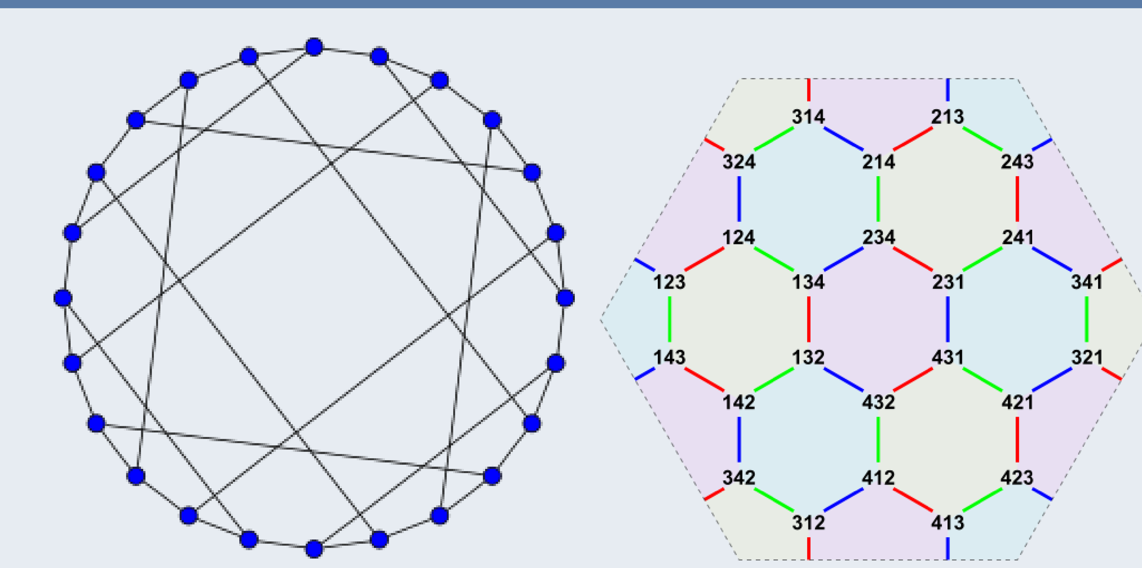
Examples of Toroidal Cubic Graphs



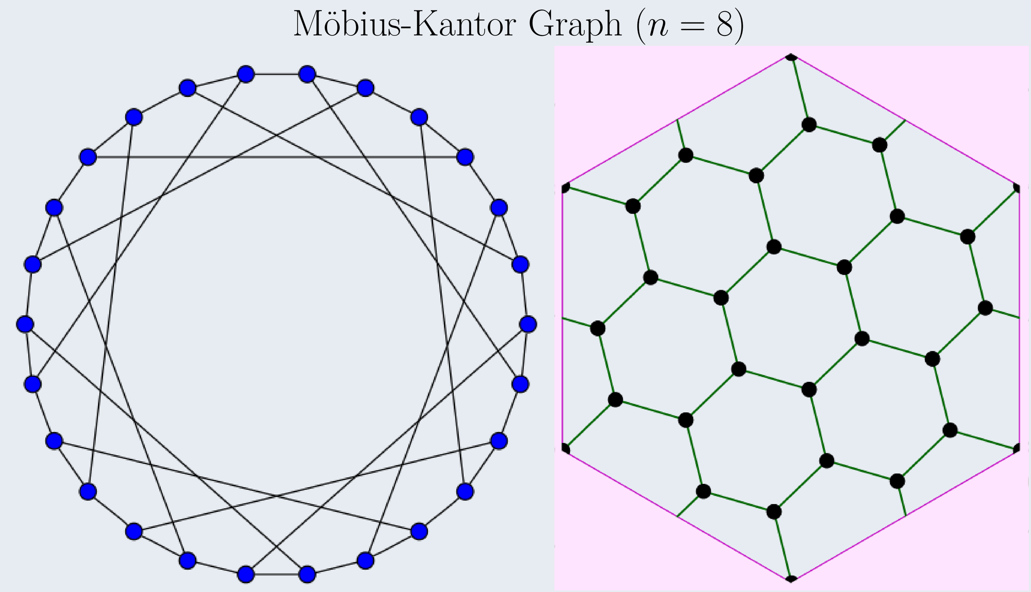
Möbius-Kantor Graph ($n = 8$)



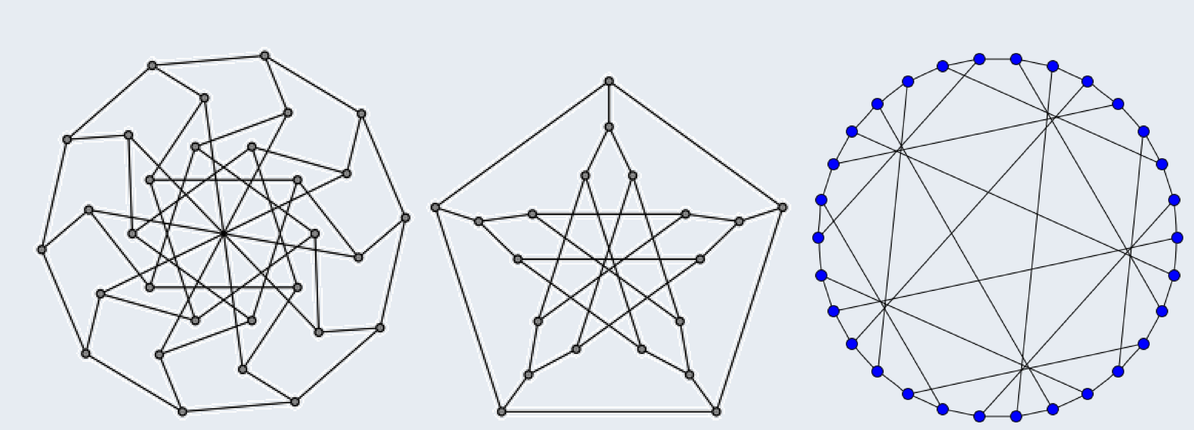
Pappus Graph ($n = 9$)



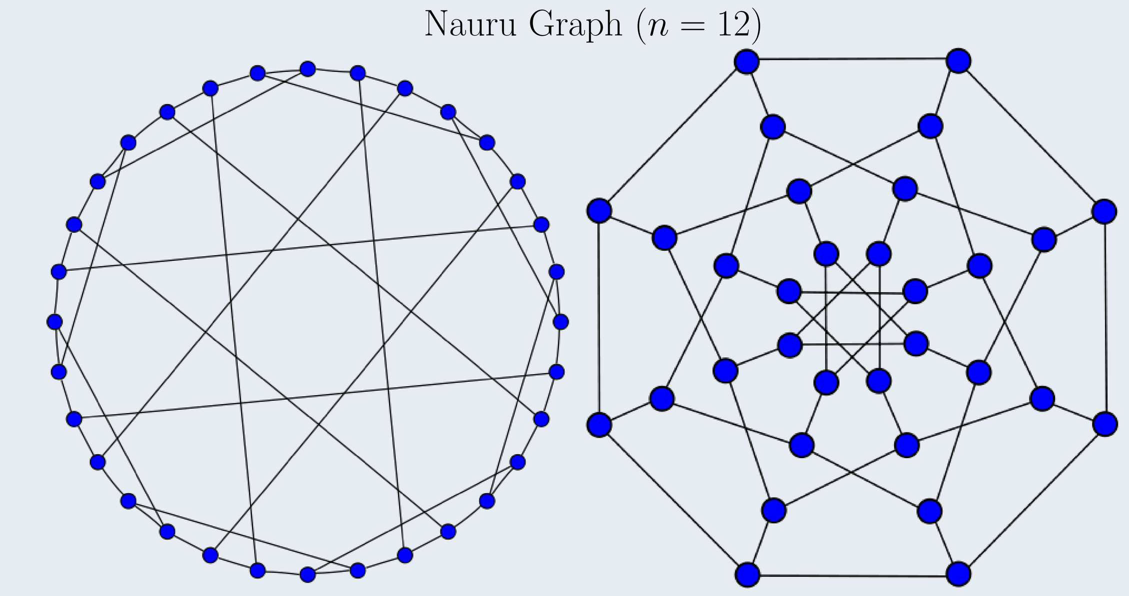
Nauru Graph ($n = 12$)



$F26A$ Graph ($n = 13$)



Tutte-Coxeter Graph ($n = 15$)



Dyck Graph ($n = 16$)